Kuwait University | Math 102 | November 9, 2008
Dept. of Math. and Comp. Science | First Exam | Duration: 90 minutes

Calculators and mobile phones are not allowed.
Answer all of the following questions.

1. Let $f(x) = \ln\left(\tan^{-1}(2x-1)\right) + 1$

   (a) Find the domain of f. (2 pts.)

   (b) Show that f has an inverse function. (2 pts.)

   (c) Find $f^{-1}(x)$ and its domain. (2 pts.)

2. (a) Prove the identity

$$2\sin^{-1}(\sqrt{x}) = \cos^{-1}(1-2x) \qquad (0 \le x \le 1)$$ (3 pts.)

   (b) Let $a > 1$. Solve the following equation for x.

$$\log_a(x+2) = \frac{2\ln x}{\ln a}$$ (2 pts.)

3. (a) Use logarithmic differentiation to find y' if

$$y = \frac{\left(\frac{1}{x-1}\right)^x \tan^{-1}x}{e^{3x}\sqrt{x^2-1}}$$ (3 pts.)

   (b) Let $y = A\sinh(mx) + B\cosh(mx)$, where A, B and m are constants. Show that

$$y'' = m^2 y$$ (2 pts.)

4. Evaluate the following integrals (3 pts. each)

   (a) $$\int \frac{dx}{(\csc x)\sqrt{4-\cos^2 x}}$$

   (b) $$\int \frac{1-e^{-2x}}{2e^{-x}} \cosh x \, dx$$

   (c) $$\int \frac{\ln x^3}{x(4+3\ln^2 x)} \, dx$$

Solution key - First midterm - CALCULUS B (Math. 102)

1. **(a)** $D_f = \{x | \tan^{-1}(2x-1) > 0\} = (\frac{1}{2}, \infty)$.

(b) $f'(x) = \frac{1}{\tan^{-1}(2x-1)} \cdot \frac{2}{1+(2x-1)^2} > 0 \ \forall x \in D_f \Longrightarrow f \nearrow \Longrightarrow f^{-1}$ exists.

(c) $y = \ln(\tan^{-1}(2x-1)) + 1 \Longleftrightarrow \tan(e^{(y-1)}) = 2x - 1 \Longleftrightarrow x = \frac{1+\tan(e^{(y-1)})}{2}$
$\Longrightarrow f^{-1}(x) = \frac{1}{2}\Big(1 + \tan(e^{(x-1)})\Big)$.

2. **(a)** Let $f(x) = 2\sin^{-1}(\sqrt{x}) - \cos^{-1}(1-2x)$

$$\Longrightarrow f'(x) = \frac{2}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} - \frac{(-1)\cdot(-2)}{\sqrt{1-(1-2x)^2}} = \frac{1}{\sqrt{x}\cdot\sqrt{1-x}} - \frac{1}{\sqrt{x}\cdot\sqrt{1-x}} = 0$$

Thus $f(x) = C$, where C is a constant. For $x = 0$, $f(0) = 2\sin^{-1}(0) - \cos^{-1}(1) = 0 = C$. Therefore, $2\sin^{-1}(\sqrt{x}) = \cos^{-1}(1-2x)$.

(b) $\log_a(x+2) = \frac{\ln x^2}{\ln a} \Longleftrightarrow \log_a(x+2) = \log_a(x^2) \Longleftrightarrow x + 2 = x^2 \Longleftrightarrow x^2 - x - 2 = 0 \Longleftrightarrow (x-2)(x+1) = 0$. Since $x > 0$, $x = 2$.

3. **(a)**

$$\ln y = -x\ln(x-1) + \ln(\tan^{-1}(x)) - 3x - \frac{1}{2}\ln(x^2-1).$$
$$\frac{1}{y}y' = -\ln(x-1) - \frac{x}{x-1} + \frac{1}{\tan^{-1}(x)} \cdot \frac{1}{1+x^2} - 3 - \frac{1}{2} \cdot \frac{2x}{x^2-1}.$$
$$\Longrightarrow y' = \frac{\left(\frac{1}{x-1}\right)^x \tan^{-1}x}{e^{3x}\sqrt{x^2-1}} \cdot \left[-\ln(x-1) - \frac{x}{x-1} + \frac{1}{(1+x^2)\tan^{-1}(x)} - 3 - \frac{x}{x^2-1}\right]$$

(b) $y' = m\,A\cosh(mx) + m\,B\sinh(mx)$, and

$$y'' = m^2\,A\sinh(mx) + m^2\,B\cosh(mx) = m^2\,y$$

4. **(a)** Let $u = \cos x \Rightarrow du = -\sin x\,dx$. Thus,

$$I = \int \frac{-du}{\sqrt{4-u^2}} = -\sin^{-1}\left(\frac{\cos x}{2}\right) + C$$

(b)

$$I = \int \frac{e^x}{e^x} \cdot \frac{1-e^{-2x}}{2e^{-x}} \cosh(\mathrm{x})\,dx = \int \frac{e^x - e^{-x}}{2}\cosh(x)\,dx$$
$$= \int \sinh(x)\cosh(x)\,dx = \frac{\sinh^2(x)}{2} + C$$

(c) Let $u = 4 + 3\ln^2 x \Rightarrow du = \frac{6\ln x}{x}dx$. Thus

$$I = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|4 + 3\ln^2 x| + C$$